Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

European Journal of Combinatorics 27 (2006) 342–363

European Journal of Combinatorics

www.elsevier.com/locate/ejc

Two oiseau decompositions of permutations and their application to Eulerian calculus

Dominique Foata[a], Arthur Randrianarivony[b]

[a] *Institut Lothaire, 1, rue Murner, F-67000 Strasbourg, France*
[b] *Faculté des Sciences, Université d'Antananarivo, BP 566 Antananarivo, Madagascar*

Received 1 December 2004; accepted 4 January 2005
Available online 19 February 2005

Abstract

Two transformations are constructed that map the permutation group onto a well-defined subset of a partially commutative monoid generated by the so-called oiseaux. Those transformations are then used to show that some bivariable statistics introduced by Babson and Steingrímsson are Euler–Mahonian.

1. Introduction

Everybody knows that each permutation of a finite set can be uniquely expressed as a product of disjoint cycles, except for changes in the order of the factors, so that the rule of commutation is the property of being disjoint. In what follows the cycles will be replaced by the so-called *oiseaux*, and those oiseaux, although they may also be regarded as disjoint, are restricted to commuting according to another rule.

By *oiseau* we mean a word, whose letters are *distinct nonnegative* integers, of the form $pdth$, where p and t are single letters, d is a (strictly) decreasing (possibly empty) word and h a (strictly) increasing (possibly empty) word; it is further assumed that p (resp. t) is the greatest (resp. the smallest) letter in $pdth$. By convention, the letter p could also be

E-mail addresses: foata@math.u-strasbg.fr (D. Foata), arthur@univ-antananarivo.mg (A. Randrianarivony).

0195-6698/$ - see front matter
doi:10.1016/j.ejc.2005.01.002

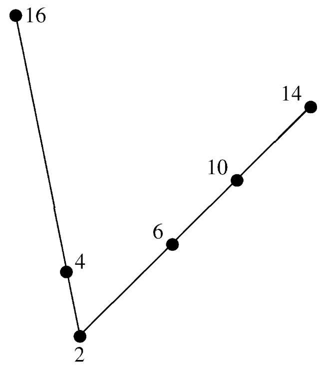

Fig. 1. An oiseau $pdth = 16, 4, 2, 6, 10, 14$.

equal to ∞. The letter "p" stands for *peak*, the letter "d" for *double-descent*, the letter "t" for *trough* and "h" for *double-rise*, or *hill*.

Remark. As shown in Fig. 1 a *oiseau* looks like a bird seen from some distance away, whose left wing is higher than the right one. The terminology was suggested by Gérard Duchamp [6], who has used these oiseaux as particular classes of descents occurring in the Dynkin Lie projector. In this example $p = 16$ is the peak of the oiseau, $d = 4$ is the double-descent factor, $t = 2$ is the trough, $h = 6, 10, 14$ is the double-rise factor.

Take the set of all the *oiseaux* as an alphabet A, supposed to be totally ordered by the lexicographic order, denoted by "$\le$", induced by the usual order on the integers. As an element of alphabet A, each oiseau will be written as $(pdth)$ with parentheses. Let $(pdth)$, $(p'd't'h')$ be two *distinct* oiseaux; we say that $(pdth)$ is *below* $(p'd't'h')$ and write $(pdth) \ll (p'd't'h')$ if and only if every letter in the word $pdth$ is less than every letter in $p'd't'h'$. We also say that $(p'd't'h')$ is *above* $(pdth)$ and write $(p'd't'h') \gg (pdth)$. Two oiseaux are said to *commute* if and only if one is below the other. One of the goals of this paper is to show that each permutation can be written as a juxtaposition product of oiseaux subjected to the previous commutation rule. As a matter of fact, two transformations **d** and **D** will be given that map permutations onto such juxtaposition products.

For instance, consider the following permutation:

$$\sigma = \overset{\wedge}{18}, \mathit{3}, \underset{\vee}{1}, \overset{\wedge}{16}, \mathit{4}, \underset{\vee}{2}, \mathbf{6}, \overset{\wedge}{9}, \underset{\vee}{5}, \mathbf{7}, \mathbf{10}, \overset{\wedge}{11}, \underset{\vee}{8}, \overset{\wedge}{13}, \underset{\vee}{12}, \mathbf{14}, \overset{\wedge}{17}, \underset{\vee}{15}, \mathbf{19},$$

where the *peaks* (resp. the *troughs*) are caused to materialize by "$\wedge$" (resp. "$\vee$") and the *double-rises* (resp. the *double-descents*) are written in bold-face (resp. in italic). Under the *first* transformation **d** the permutation is mapped onto a *oiseau-word* v, i.e., a word whose letters are oiseaux of the form $(p_0 d_0 t_0 h_0) \ldots (p_r d_r t_r h_r)$:

$$v = (\infty, 0, \mathbf{19})(\overset{\wedge}{18}, \mathit{3}, \underset{\vee}{1})(\overset{\wedge}{16}, \mathit{4}, \underset{\vee}{2}, \mathbf{6}, \mathbf{10}, \mathbf{14})(\overset{\wedge}{9}, \underset{\vee}{5}, \mathbf{7})(\overset{\wedge}{11}, \underset{\vee}{8})(\overset{\wedge}{13}, \underset{\vee}{12})(\overset{\wedge}{17}, \underset{\vee}{15}).$$

Using the commutation rule on oiseaux, this juxtaposition product can be rewritten as

$$v' = (\infty, 0, \mathbf{19})(\overset{\wedge}{18}, \mathit{3}, \underset{\vee}{1})(\overset{\wedge}{16}, \mathit{4}, \underset{\vee}{2}, \mathbf{6}, \mathbf{10}, \mathbf{14})(\overset{\wedge}{17}, \underset{\vee}{15})(\overset{\wedge}{13}, \underset{\vee}{12})(\overset{\wedge}{9}, \underset{\vee}{5}, \mathbf{7})(\overset{\wedge}{11}, \underset{\vee}{8}).$$

Finally, the permutation σ' that is sent over v' under the *second* transformation **D** reads

$$\sigma' = \overset{\wedge}{19}, \underset{\vee}{1}, \overset{\wedge}{18}, \mathit{3}, \underset{\vee}{2}, \mathbf{6}, \mathbf{10}, \mathbf{14}, \overset{\wedge}{16}, \underset{\vee}{15}, \overset{\wedge}{17}, \underset{\vee}{12}, \overset{\wedge}{13}, \underset{\vee}{5}, \mathbf{7}, \overset{\wedge}{9}, \underset{\vee}{8}, \overset{\wedge}{11}, \mathit{4}.$$

Notice that the *nonzero troughs* in the permutations σ, σ' and the words v, v' are the same: 1, 2, 5, 8, 12, 15; as are the *double-descents* (written in italic): *3, 4*. As seen in the following theorem, the transformations **d**, **D** and also the commutation rule on oiseaux *preserve* the *set* of all the nonzero troughs and double-descents.

Let $\sigma = x_1 x_2 \ldots x_n$ be a permutation of $1, 2, \ldots, n$. Following Babson–Steingrímsson [1], $(b - ca)\sigma$ designates the number of triples (i, j, k) such that $1 \le i < j < k = j + 1 \le n$ and $x_j > x_i > x_k$. When the parentheses are removed from a oiseau-word, we get another word, say $\overline{v}$, whose letters are nonnegative integers. It then makes sense to define $(b - ca)v := (b - ca)\overline{v}$. The main result of the paper can be stated as follows.

Theorem 1.1. *For each positive integer n there exist two sets $B_n^{\min}$ and $B_n^{\max}$ of oiseau-words having the following properties:*

(i) *the transformation* **d** *is a bijection of the permutation group $\mathcal{S}_n$ onto $B_n^{\min}$ with the property*

$$(b - ca)\sigma = (b - ca)\mathbf{d}\,\sigma;$$

(ii) *there is a bijection θ of $B_n^{\min}$ onto $B_n^{\max}$ such that*

$$(b - ca)v = (b - ca)\theta(v);$$

(iii) *the transformation* **D** *is a bijection of $\mathcal{S}_n$ onto $B_n^{\max}$ with the property*

$$(b - ca)\sigma = (b - ca)\mathbf{D}\,\sigma + n - \mathrm{L}\,\sigma - \mathrm{des}\,\sigma,$$

where $\mathrm{L}\,\sigma$ is the rightmost letter of σ and $\mathrm{des}\,\sigma$ the usual number of descents of the permutation;

(iv) *the bijections* **d**, θ, **D** *preserve the set of the nonzero troughs and double-descents.*

Going back to the previous numerical example we can verify that in σ the letter 3 is to the left of 16, 4, 2; 6 is to the left of 9, 5; the letters 9, 10 are on the left of 11, 8; 16 is on the left of 17, 15; so $(b - ca)\sigma = 5$. Also $(b - ca)v = (b - ca)v' = 5$. However, $(b - ca)\sigma' = 12$, $\mathrm{L}\,\sigma' = 4$, $\mathrm{des}\,\sigma' = 8$ and $n = 19$. As $v' = \mathbf{D}\,\sigma'$, we have verified that

$$12 = (b - ca)\sigma' = (b - ca)\mathbf{D}\,\sigma' + n - \mathrm{L}\,\sigma' - \mathrm{des}\,\sigma' = 5 + 19 - 4 - 8,$$

so (iii) holds.

The composition product $\Phi := \mathbf{D}^{-1}\theta\mathbf{d}$ is then a bijection of $\mathcal{S}_n$ onto itself with the property

$$(b - ca)\,\Phi(\sigma) = (b - ca)\sigma + n - \mathrm{L}\;\Phi(\sigma) - \mathrm{des}\;\Phi(\sigma) \tag{1.1}$$

and this latter result will be the basic ingredient for proving that two bivariable statistics are Euler–Mahonian.

(1)	(2)	(3)	(4)	(5)	(6)	(7)
σ $\mathcal{S}_4$	$\mathbf{d}\,\sigma$ $B_4^{\min}$	$\theta\mathbf{d}\,\sigma{=}\mathbf{D}\,\sigma'$ $B_4^{\max}$	$\mathbf{D}^{-1}\theta\mathbf{d}\,\sigma$ $=\Phi(\sigma)=:\sigma'$	$(b-ca)\sigma'$	$(b-ca)\sigma$	$f(\sigma')$
1234	(∞01234)	id	1234	0	0	0
124**3**	(∞012)(4**3**)	id	124**3**	0	0	0
13**2**4	(∞014)(3**2**)	id	14**2**3	0	0	0
134**2**	(∞013)(4**2**)	id	13**2**4	0	1	−1
14**2**3	(∞01)(4**2**3)	id	134**2**	1	0	1
14**32**	(∞01)(4**32**)	id	14**32**	0	0	0
2**1**34	(∞034)(2**1**)	id	34**1**2	1	0	1
2**1**4**3**	(∞0)(2**1**)(4**3**)	(∞0)(4**3**)(2**1**)	4**31**2	0	0	0
23**1**4	(∞024)(3**1**)	id	24**1**3	1	1	0
234**1**	(∞023)(4**1**)	id	23**1**4	1	2	−1
24**1**3	(∞02)(4**1**3)	id	2**1**34	0	1	−1
24**31**	(∞02)(4**31**)	id	2**1**4**3**	0	1	−1
3**1**24	(∞04)(3**1**2)	id	4**1**23	0	0	0
3**1**4**2**	(∞0)(3**1**)(4**2**)	id	3**2**4**1**	2	1	1
3**21**4	(∞04)(3**21**)	id	4**1**3**2**	0	0	0
3**2**4**1**	(∞0)(3**2**)(4**1**)	id	3**21**4	0	2	−2
34**1**2	(∞03)(4**1**2)	id	3**1**24	0	1	−1
34**21**	(∞03)(4**21**)	id	3**1**4**2**	1	1	0
4**1**23	(∞0)(4**1**23)	id	234**1**	2	0	2
4**1**3**2**	(∞0)(4**1**)(3**2**)	id	4**2**3**1**	1	0	1
4**21**3	(∞0)(4**21**3)	id	34**21**	1	0	1
4**2**3**1**	(∞0)(4**2**)(3**1**)	id	4**21**3	0	1	−1
4**31**2	(∞0)(4**31**2)	id	24**31**	1	0	1
4**321**	(∞0)(4**321**)	id	4**321**	0	0	0

Fig. 2.

In Fig. 2 the permutations in $\mathcal{S}_4$ are listed in column (1). The second column contains the images of each permutation σ under the bijection $\mathbf{d}$. We can then read all the oiseau-words belonging to $B_4^{\min}$. In column (3) the elements $\theta\mathbf{d}\,\sigma$ that are listed are the oiseau-words belonging to $B_4^{\max}$. The entry "id" means that $\mathbf{d}\,\sigma = \theta\mathbf{d}\,\sigma$. This is the case for all the oiseau-words except for $\sigma = 2\mathbf{143}$. Notice that the oiseaux $(2\mathbf{1})$ and $(4\mathbf{3})$ commute. The images $\sigma' := \Phi(\sigma)$ are listed in column (4). In columns (5) and (6) the statistics $(b-ca)$ are calculated for σ' and for σ, respectively. The function f in column (7) is defined to be $f(\sigma') := n - \mathrm{L}\,\sigma' - \operatorname{des}\sigma'$, so (1.1) reads $(b-ca)\sigma' = (b-ca)\sigma + f(\sigma')$. To verify that (1.1) holds for $n = 4$ we just have to notice that column (5) $=$ column (6) $+$ column (7). The nonzero troughs and the double-descents are reproduced in bold-face, so statement (iv) of Theorem 1.1 is illustrated by the fact that in each row the *set* of letters in bold-face is the same in columns (1), (2), (3) and (4). For instance, **2** is a *trough* in $\sigma = 3\mathbf{2}4\mathbf{1}$ and $\mathbf{d}\,\sigma = (\infty 0)(3\mathbf{2})(4\mathbf{1})$, but a *double-descent* in $\Phi(\sigma) = 3\mathbf{21}4$.

For constructing the transformations $\mathbf{d}$, $\mathbf{D}$ and θ, and accordingly defining the two sets $B_n^{\min}$ and $B_n^{\max}$, we have recourse to the theory of *partially commutative monoids*, as developed by Cartier and Foata [4]. The salient features are recalled in Section 2. The partially commutative monoid generated by the set of the *oiseaux* is then described in Section 3. The next two sections are devoted to the constructions of the transformations $\mathbf{d}$

and **D**. Surprisingly, part (iii) of Theorem 1.1 requires a delicate analysis that is carried out in Section 6. The final sections deal with Eulerian Calculus. In particular, we prove that four conjectures made by Babson and Steingrímsson [1] are correct.

The following notation will be used throughout. Let A be a nonempty set; we let A^* denote the free monoid generated by A, that is, the set of all the finite words $w = a_1a_2\dots a_m$ whose letters a_i belong to A. The number m of the letters in the word w is the *length* of w, denoted by $|w|$. If E is a subset of A, then $|w|_E$ is the number of letters of w that belong to E. When the underlying alphabet A is the set of the integers, then $\operatorname{tot} w$ designates the *sum* of its letters: $\operatorname{tot} w = a_1 + a_2 + \cdots + a_m$ (“tot” stands for “total”). As noted before, $\mathrm{L}\, w$ designates the rightmost letter a_m of w. Furthermore, $\mathrm{F}\, w$ refers to its leftmost letter a_1 (the first one).

When the alphabet A is totally ordered, the number of *descents* of w is defined to be the number of integers j such that $1 \le j \le m-1$ and $a_j > a_{j+1}$. It is expressed as $(ba)w$ or $\operatorname{des} w$. Further statistics, that are introduced in the final sections, refer to Eulerian Calculus proper.

2. Partially commutative monoids

Let A^* be the free monoid generated by a nonempty set A and let C be a subset of $A \times A$ containing no element of the form (a, a) and such that (a', a) belongs to C if (a, a') does. Two words w and w' are said to be C-*adjacent* if there exist two words u and v and an ordered pair (a, a') in C such that $w = uaa'v$ and $w' = ua'av$. Two words w and w' are said to be C-*equivalent* if they are equal, or if there exists a sequence of words $w_0, w_1, \dots, w_p$ such that $w_0 = w$, $w_p = w'$ and w_{i-1} and w_i are C-adjacent for $1 \le i \le p$. This defines an equivalence relation R_C on A^*, compatible with the multiplication in A^*. The quotient monoid A^*/R_C is denoted by $L(A; C)$ and is called the C-*partially commutative monoid generated by* A. The C-equivalence class of a letter $a \in A$ is denoted by $[a]$. Two letters a, a' are said to *commute* if $[a][a'] = [a'][a]$ in $L(A; C)$.

A subset F of A is said to be *commutative* if it is finite, nonempty and if any two of its elements commute. For such a subset let $[F] := \prod_{a\in A}[a]$. A letter a is said to be *linked to* a subset F of A if $a \in F$ or if there exists a letter in F that does not commute with a. Let F, F' be two subsets of A; then, F is said to be *contiguous* to F' if every letter of F' is linked to F. Each sequence $(F_1, \dots, F_r)$ of commutative subsets of A is called a V-*sequence* if for each $i = 1, \dots, r-1$ the subset F_i is contiguous to F_{i+1}.

Theorem 2.1. *For every element* u *in* $L(A; C)$ *there is a unique* V*-sequence* $(F_1, \dots, F_r)$ *such that* $u = [F_1]\cdots[F_r]$.

(See [4, p. 11] for a proof.)

We mention a further property that is needed in the present context. Suppose that the alphabet A is totally ordered and let “$\le$” denote the total ordering. A word $w = a_1a_2\dots a_m$ is said to be C-*minimal* (resp. C-*maximal*) if for each $i = 1, 2, \dots, m-1$ the following property holds:

$$\text{if } a_i \text{ and } a_{i+1} \text{ commute, then } a_i < a_{i+1} \text{ (resp. } a_i > a_{i+1}). \tag{2.1}$$

Proposition 2.2. *Each C-equivalence class in $L(A;C)$ contains one and only one C-minimal (resp. C-maximal) word.*

Proof. Let $u = [b_1][b_2]\ldots[b_m]$ be a C-equivalence class and let $(F_1, F_2, \ldots, F_r)$ be its V-sequence. Denote the minimum of F_1 (with respect to the total ordering on A) by a_1. The product $[F_1 \setminus \{a_1\}][F_2]\cdots[F_r]$ in $L(A;C)$ admits a V-sequence $(G_1, G_2, \ldots, G_s)$. More essentially, the inclusion $F_1 \setminus \{a_1\} \subset G_1$ holds, as can be verified directly, or by using Corollary 1.4 on p. 15 in [4]. Extract $a_2 := \min G_1$ from B_1. If a_2 belongs to F_1, both a_1 and a_2 commute and $a_1 < a_2$; if $a_2 \notin F_1$, then a_2 necessarily belongs to F_2 and commutes with all the letters in $F_1 \setminus \{a_1\}$. Consequently, a_1 and a_2 do not commute. Again form the new class $[G_1 \setminus \{a_2\}][G_2]\cdots[G_s]$, and so on. At the end we get a word $a_1a_2\ldots a_m$ such that either $a_i < a_{i+1}$, or a_i and a_{i+1} do not commute.

To prove the uniqueness, take two *distinct* C-minimal words v and v' supposed to be C-equivalent. It suffices to consider the case where $v = a_1a_2\ldots a_m$, $v' = a_{i_1}a_{i_2}\ldots a_{i_m}$ with $a_1 \neq a_{i_1}$. Those words may also be expressed as $v = a_1a_2\ldots a_ja_{i_1}a_{j+2}\ldots a_m$ and $v' = a_{i_1}a_{i_2}\ldots a_{i_k}a_1a_{i_{k+2}}\ldots a_{i_m}$, with $a_{i_1} \neq a_1, a_2, \ldots, a_j$ and $a_1 \neq a_{i_1}, a_{i_2}, \ldots, a_{i_k}$. As v and v' are supposed to be C-equivalent, the letter a_{i_1} (resp. a_1) commutes with $a_1, a_1, \ldots, a_j$ (resp. with $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$). But as v and v' are both C-minimal, the two relations $a_1 < a_{i_1}$, $a_{i_1} < a_1$ hold, a contradiction. To prove the "resp." part, simply replace "min" vy "max" and "$<$" by "$>$" in the above derivation. □

In the following the map that sends each C-minimal word in A^* onto the C-maximal word that belongs to the same C-equivalence class will be denoted by θ. If $v = a_1a_2\ldots a_m$ is C-minimal, we form the C-class $u = [a_1][a_2]\cdots[a_m]$, then its V-sequence $(F_1, F_2, \ldots, F_r)$. Following the proof of the previous proposition, define $b_1 := \max F_1$ and form the V-sequence $(G_1, G_2, \ldots, F_s)$ of $[F_1 \setminus \{b_1\}][F_2]\ldots[F_r]$. Further, define $b_2 := \max G_1$, and so on. Finally, $\theta(v) = b_1b_2\ldots b_m$.

3. The oiseau partially commutative monoid

Now A will designate the set of all the oiseaux $(pdth)$, as defined in the introduction and C will be the set of all pairs $((pdth), (p'd't'h'))$ of oiseaux such that either $(pdth) \ll (p'd't'h')$ or $(pdth) \gg (p'd't'h')$. We form the free monoid A^* generated by A. Its elements are the *oiseau-words* $v = (p_0d_0t_0h_0)(p_1d_1t_1h_1)\ldots(p_rd_rt_rh_r)$. We also consider the C-partially commutative monoid $L(A;C)$. The C-equivalence class of each oiseau $(pdth)$ will be denoted by $[pdth]$.

For each $n \geq 1$ the two classes of oiseau-words $B_n^{\min}$ and $B_n^{\max}$, mentioned in the statement of Theorem 1.1, are defined as follows. A oiseau-word $v = (p_0d_0t_0h_0)(p_1d_1t_1h_1)\ldots(p_rd_rt_rh_r)$ belongs to $B_n^{\min}$ (resp. to $B_n^{\max}$) if and only if the following three conditions hold:

(i) the oiseau-word v is C-minimal (resp. C-maximal);
(ii) $p_0 = \infty$, $d_0 = e$ (the empty word) and $t_0 = 0$;
(iii) the factor $h_0p_1d_1t_1h_1\ldots p_rd_rt_rh_r$ is a permutation of $1, 2, \ldots n$.

Let $[v] = [p_0d_0t_0h_0]\ldots[p_rd_rt_rh_r]$ be the C-equivalence class of an element $v = (p_0d_0t_0h_0)\ldots(p_rd_rt_rh_r)$ of $B_n^{\min}$ (resp. of $B_n^{\max}$). As $p_0d_0t_0 = \infty 0$, no oiseau

$(p_i d_i t_i h_i)(i = 1, \ldots, r)$ can commute with $(p_0 d_0 t_0 h_0) = (\infty 0 h_0)$. Hence, all the words in the C-equivalence class $[v]$ start with $(\infty 0 h_0)$ and the first factor in the V-sequence of $[v]$ is reduced to the single term $[(\infty 0 h_0)]$. The set of all the C-equivalence classes that contain one element of $B_n^{\min}$ (and only one by Proposition 2.2) will be designated by B_n. The three sets B_n, $B_n^{\min}$, $B_n^{\max}$ can be put into a one-to-one correspondence. In particular, $B_n^{\max} = \theta(B_n^{\min})$, where θ is the bijection introduced at the end of Section 2.

Example. The following word:

$$v = (\infty, 0, 19)(18, 3, 1)(16, 4, 2, 6, 10, 14)(9, 5, 7)(11, 8)(13, 12)(17, 15)$$

is an element of $B_{19}^{\min}$. To derive $\theta(v)$, calculate the V-sequence of $[v]$ that reads $(F_1, F_2, F_3, F_4, F_5)$ with

$$F_1 = \{(\infty, 0, 19)\}, \qquad F_2 = \{(18, 3, 1)\}, \qquad F_3 = \{(16, 4, 2, 6, 10, 14)\},$$
$$F_4 = \{(9, 5, 7), (13, 12), (17, 15)\}, \qquad F_5 = \{(11, 8)\}.$$

Then $a_1 = \max F_1 = (\infty, 0, 19)$. The V-sequence of $[F_2][F_3][F_4][F_5]$ is simply (F_2, F_3, F_4, F_5), so $a_2 = (18, 3, 1)$. The V-sequence of $[F_3][F_4][F_5]$ is (F_3, F_4, F_5) and $a_3 = (16, 4, 2, 6, 10, 14)$. The V-sequence of $[F_4][F_5]$ is (F_4, F_5) and $a_4 = \max F_4 = (17, 15)$. The V-sequence of $[F_4 \setminus \{a_4\}][F_5]$ is $(\{(9, 5, 7), (13, 12)\}, \{(11, 8)\})$ and $a_5 = (13, 12)$. Finally, $a_6 = (9, 5, 7)$, $a_7 = (11, 8)$, so

$$\theta(v) = (\infty, 0, 19)(18, 3, 1)(16, 4, 2, 6, 10, 14)(17, 15)(13, 12)(9, 5, 7)(11, 8).$$

The next two sections will be devoted to constructing two bijections $\mathbf{d}$ and $\mathbf{D}$ of the permutation group $\mathcal{S}_n$ onto $B_n^{\min}$ and $B_n^{\max}$, respectively.

4. The first transformation

To construct the first bijection $\mathbf{d}$ of $\mathcal{S}_n$ onto $B_n^{\min}$ we proceed as follows. Let $\sigma = x_1 x_2 \cdots x_n$ be a permutation of the word $12 \ldots n$, written as a linear word. Let $x_0 := 0$, $x_{-1} = x_{n+1} := +\infty$ and consider the word $0\sigma = x_0 x_1 \ldots x_n$. By *peak* of 0σ we mean a letter x_j such that $1 \le j \le n$ and $x_{j-1} < x_j$, $x_j > x_{j+1}$. By *trough* we mean a letter x_j such that $0 \le j \le n$ and $x_{j-1} > x_j$, $x_j < x_{j+1}$. Accordingly, each permutation 0σ has $(r+1)$ troughs and r peaks for some $r \ge 0$. By *double-descent* we mean a letter x_j such that $1 \le j \le n-1$ and $x_{j-1} > x_j > x_{j+1}$. By *double-rise* we mean a letter x_j such that $1 \le j \le n$ and $x_{j-1} < x_j < x_{j+1}$. As above, p and t will designate *letters* which are peaks and troughs, respectively, while d, h, will designate *words* all letters which are double-descents, double-rises, respectively. There is a unique factorization

$$t_0 h_0 \mid p_1 d_1 t_1 h_1 \mid p_2 d_2 t_2 h_2 \mid \ldots \mid p_r d_r t_r h_r \tag{4.1}$$

of 0σ, called its *peak factorization*, having the following properties:

(1) $r \ge 0$; $t_0 = 0$;
(2) the p_i's are the *peaks* and the t_i's the *troughs* of the permutation;
(3) for each i the symbols h_i, d_i are *words*, possibly empty, all letters of which are *double-rises*, *double-descents*, respectively. It will be convenient to let $p_0 = p_{r+1} := +\infty$.

In each component $p_j d_j t_j h_j$ the peak p_j is not necessarily the greatest letter of the factor, so $(p_j d_j t_j h_j)$ may not be a oiseau. The transformation $\mathbf{d}$ will consist of moving *to the left* the different *double-rises* in such a way that the new factors, say, $(p_j d_j t_j h'_j)$ will be oiseaux; in particular, $p_j \gg h'_j$.

Let y be a *double-rise* of 0σ, so that y is a letter of the factor h_j for some j $(0 \le j \le r)$. Define $\varphi(y)$ to be the *greatest integer* i such that $0 \le i \le j$ and $p_i > y > t_i$. Then, the following inequalities hold:

$$p_i > y > t_i, y > p_{i+1}, \ldots, y > p_j, t_j < y < p_{j+1}. \tag{4.2}$$

Next, define h'_i to be the *increasing word* of all double-rises y in σ such that $\varphi(y) = i$. If $\varphi(y) \ne i$ for every double-descent y of 0σ, let h'_i be the empty word. Clearly, $(\infty 0 h'_0)$ is a oiseau, as is each letter $(p_i d_i t_i h'_i)$ for $i = 1, 2, \ldots, r$. Define $\mathbf{d}\,\sigma$ to be the following element in A^* (remember that $t_0 = 0$):

$$\mathbf{d}\,\sigma := (\infty t_0 h'_0)(p_1 d_1 t_1 h'_1)(p_2 d_2 t_2 h'_2) \ldots (p_r d_r t_r h'_r). \tag{4.3}$$

For example, start with the peak-factorized permutation

$$0\sigma = \underset{0}{0} \mid \underset{1}{\overset{\wedge}{18}, 3, \underset{\vee}{1}} \mid \underset{2}{\overset{\wedge}{16}, 4, \underset{\vee}{2}, \mathbf{6}} \mid \underset{3}{\overset{\wedge}{9}, \underset{\vee}{5}, \mathbf{7}, \mathbf{10}} \mid \underset{4}{\overset{\wedge}{11}, \underset{\vee}{8}} \mid \underset{5}{\overset{\wedge}{13}, \underset{\vee}{12}, \mathbf{14}} \mid \underset{6}{\overset{\wedge}{17}, \underset{\vee}{15}, \mathbf{19}},$$

that has seven factors, numbered from 0 to 6. The peaks (resp. the troughs) are caused to materialize by "$\wedge$" (resp. "$\vee$") and the double-rises are written in bold-face. We have $p_0 = \infty$, $p_1 = 18$, $p_2 = 16$, $p_3 = 9$, $p_4 = 11$, $p_5 = 13$, $p_6 = 17$. Also, $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, $t_3 = 5$, $t_4 = 8$, $t_5 = 12$, $t_6 = 15$. Next, the double-descent factors read $d_1 = 3$; $d_2 = 4$, $d_3 = d_4 = d_5 = d_6 = e$ (the empty word) and the double-rise factors $h_0 = h_1 = e$; $h_2 = \mathbf{6}$; $h_3 = \mathbf{7, 10}$; $h_4 = e$; $h_5 = \mathbf{14}$; $h_6 = \mathbf{19}$.

Thus $\varphi(\mathbf{6}) = 2$, $\varphi(\mathbf{7}) = 3$, $\varphi(\mathbf{10}) = 2$, $\varphi(\mathbf{14}) = 2$, $\varphi(\mathbf{19}) = 0$ (remember that $p_0 = \infty$) and $h'_0 = \mathbf{19}$; $h'_2 = \mathbf{6, 10, 14}$; $h'_3 = \mathbf{7}$; $h'_4 = h'_5 = h'_6 = e$. Hence

$$\mathbf{d}\,\sigma = (\infty, 0, \mathbf{19})(18, 3, 1)(16, 4, 2, \mathbf{6, 10, 14})(9, 5, \mathbf{7})(11, 8)(13, 12)(17, 15).$$

Go back to the general case. For each $i = 0, 1, \ldots, r - 1$ the trough t_i is less than both p_i and p_{i+1}. Hence, if $p_i > p_{i+1}$, the two oiseaux $(p_i d_i t_i h'_i)$ and $(p_{i+1} d_{i+1} t_{i+1} h'_{i+1})$ cannot commute. In an equivalent manner, if $(p_i d_i t_i h'_i)$ and $(p_{i+1} d_{i+1} t_{i+1} h'_{i+1})$ commute, then $p_i < p_{i+1}$ and also $(p_i d_i t_i h'_i) < (p_{i+1} d_{i+1} t_{i+1} h'_{i+1})$ (for the lexicographic order). But as the oiseaux commute, the $<$-sign can be replaced by the $\ll$-sign, so $\mathbf{d}\,\sigma$ is *minimal*.

The map $\mathbf{d}$ *is bijective*. The construction of $\mathbf{d}$ given above is perfectly reversible. Let $v = (\infty t_0 h'_0)(p_1 d_1 t_1 h'_1)(p_2 d_2 t_2 h'_2) \ldots (p_r d_r t_r h'_r)$ be an element of $B_n^{\min}$ and let y be a *double-rise* in the word $p_i d_i t_i h'_i$ $(0 \le i \le r)$ (with $p_0 = \infty$, $d_0 = e$ and $t_0 = 0$).

If $p_i < p_{i+1}$ (remember that $p_{r+1} = \infty$ by convention), let $\varphi^-(y) := i$. If $p_i > p_{i+1}$, define $\varphi^-(y)$ to be the smallest integer j such that $i + 1 \le j \le r$ and $y < p_{j+1}$. Then, define h_j to be the *increasing word* of all double-rises y from each of the words $p_0 d_0 t_0 h'_0, \ldots, p_r d_r t_r h'_r$ such that $\varphi^-(y) = j$ and form the permutation $\sigma^- := t_0 h_0 p_1 d_1 t_1 h_1 p_r d_r t_r h_r$.

As the inequalities in (4.2) hold, we have the equivalence

$$[y \text{ is a letter of } h_j \text{ and } \varphi(y) = i] \Leftrightarrow [y \text{ is a letter of } h'_i \text{ and } \varphi^-(y) = j].$$

To show that $\mathbf{d}$ is reversible we only have to verify that the factorization

$$t_0h_0 \mid p_1d_1t_1h_1 \mid \ldots \mid p_rd_rt_rh_r,$$

is the peak factorization of σ^-, essentially verifying that the p_j's are the peaks of σ^-. When $p_j > p_{j+1}$, we do not have $p_jd_jt_jh'_j \gg p_{j+1}d_{j+1}t_{j+1}h'_{j+1}$, because v is supposed to be minimal; hence $t_j < p_{j+1}$. The double-rises in h'_j less than p_{j+1} occur in h_j, while the double-rises greater than p_{j+1} occur in a factor h_i with $i \geq j+1$. It follows that p_{j+1} is greater than all the letters of h_j. It is then a peak of σ^-, so $\sigma^- = \sigma$. Thus $\mathbf{d}$ is a bijection.

Example. Start with the C-minimal word

$$\begin{array}{ccccccc} v = (\infty, 0, \mathbf{19}) & (18, 3, 1) & (16, 4, 2, \mathbf{6}, \mathbf{10}, \mathbf{14}) & (9, 5, \mathbf{7}) & (11, 8) & (13, 12) & (17, 15) \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

We have $\varphi^-(\mathbf{19}) = 6$; $\varphi^-(\mathbf{6}) = 2$; $\varphi^-(\mathbf{10}) = 3$; $\varphi^-(\mathbf{14}) = 5$; $\varphi^-(\mathbf{7}) = 3$; so $h_0 = h_1 = e$; $h_2 = \mathbf{6}$; $h_3 = \mathbf{7}, \mathbf{10}$; $h_4 = e$; $h_5 = \mathbf{14}$; $h_6 = \mathbf{19}$ and

$$\mathbf{d}^{-1}v = \sigma = \mid 18, 3, 1 \mid 16, 4, 2, \mathbf{6} \mid 9, 5, \mathbf{7}, \mathbf{10} \mid 11, 8 \mid 13, 12, \mathbf{14} \mid 17, 15, \mathbf{19}.$$

5. The second transformation

In the second transformation the key role will be played by the double-descents rather than the double-rises. The peak factorization is replaced by the *trough factorization* that consists of cutting the permutation just before each trough. More formally, each permutation $\sigma = x_1x_2\ldots x_n$ has a unique factorization

$$h_0p_0d_0 \mid t_1h_1p_1d_1 \mid t_2h_2p_2d_2 \mid \ldots \mid t_rh_rp_rd_r,$$

where all the previous notation for the letters h, p, d, t has been kept. By convention, this time, $t_0 = t_{r+1} = -\infty$.

Construction of the transformation $\mathbf{D}$. There are four steps in the construction:

(1) The first step consists of transposing the factors t_ih_i and p_id_i in each compartment, for $i = 1, 2, \ldots, r$, the first compartment remaining alike, to obtain

$$h_0p_0d_0 \mid p_1d_1t_1h_1 \mid p_2d_2t_2h_2 \mid \ldots \mid p_rd_rt_rh_r.$$

(2) Let x be a letter of the factor d_i $(1 \leq i \leq r)$. Either there exists an integer j such that $1 \leq j \leq i$ and $p_j > x > t_j$ and $\psi^-(x)$ will denote the *greatest* integer with those properties, or such an integer does not exist and we define $\psi^-(x) := 0$. Also let $\psi^-(x) := 0$ for each letter x in the factor d_0.

(3) For each $i = 0, 1, \ldots, r$ form the *decreasing* factor $\overline{d}_i$ with all the letters x such that $\psi^-(x) = i$.

(4′) If the word $\overline{d}_0$ is empty, define $\mathbf{D}\sigma$ to be the word in A^* [later it will be shown that each letter $(\ldots)$ is a oiseau]

$$\mathbf{D}\sigma := (\infty 0 h_0 p_0)(p_1\overline{d}_1t_1h_1)(p_2\overline{d}_2t_2h_2)\ldots(p_r\overline{d}_rt_rh_r).$$

(4″) If $\overline{d}_0$ is nonempty, denote its rightmost letter by t_0; hence $\overline{d}_0$ may be written as $\overline{d}_0 = \overline{d'_0}t_0$ for some (decreasing) factor $\overline{d'_0}$. If $h_0 = e$, let $h'_0 = h''_0 := e$; if $t_0 \ll h_0$, let $h'_0 := e,\ h''_0 := h_0$; if $h_0 \ll t_0$, let $h'_0 := h_0, h''_0 := e$. Otherwise, the word h_0 may be expressed (in a unique manner) as $h_0 = h'_0 h''_0$, in such a way that $h'_0 \ll t_0 \ll h''_0$. Then define

$$\mathbf{D}\,\sigma := (\infty 0 h'_0)(p_0\overline{d'_0}t_0h''_0)(p_1\overline{d}_1t_1h_1)(p_2\overline{d}_2t_2h_2)\ldots(p_r\overline{d}_rt_rh_r).$$

Example 1. Consider the following permutation already trough-factorized:

$$\sigma = 18, \overset{\wedge}{20} \mid \underset{\vee}{3}, 7, \overset{\wedge}{17} \mid \underset{\vee}{16}, \overset{\wedge}{19}, 15, 6 \mid \underset{\vee}{2}, 12, \overset{\wedge}{14}, 10 \mid \underset{\vee}{9}, \overset{\wedge}{13}, 11, 8 \mid \underset{\vee}{1}, \overset{\wedge}{5}, 4.$$
$$0 \qquad 1 \qquad 2 \qquad 3 \qquad 4 \qquad 5$$

Step (1) yields

$$\overset{\wedge}{18}, \overset{\wedge}{20} \mid \overset{\wedge}{17}, \underset{\vee}{3}, 7 \mid \overset{\wedge}{19}, 15, 6, \underset{\vee}{16} \mid \overset{\wedge}{14}, 10, \underset{\vee}{2}, 12 \mid \overset{\wedge}{13}, 11, 8, \underset{\vee}{9} \mid \overset{\wedge}{5}, 4, \underset{\vee}{1}.$$
$$0 \qquad 1 \qquad 2 \qquad 3 \qquad 4 \qquad 5$$

In step (2) we have to define: $\psi^-(15) = 1$, $\psi^-(6) = 1$, $\psi^-(10) = 3$, $\psi^-(11) = 4$, $\psi^-(8) = 3$; $\psi^-(4) = 5$, so we may form, as prescribed in step (3), $\overline{d}_0 = e$; $\overline{d}_1 = 15, 6$; $\overline{d}_2 = e$; $\overline{d}_3 = 10, 8$; $\overline{d}_4 = 11$; $\overline{d}_5 = 4$.

As $\overline{d}_0 = e$, step (4′) applies, so

$$\mathbf{D}\,\sigma := (\infty, 0, 18, 20)(17, 15, 6, 3, 7)(19, 16)(14, 10, 8, 2, 12)(13, 11, 9)(5, 4, 1).$$

Example 2. With the permutation

$$\sigma = 1, 6, 11, \overset{\wedge}{18}, 17 \mid \underset{\vee}{16}, \overset{\wedge}{20}, 15 \mid \underset{\vee}{8}, 12, \overset{\wedge}{14} \mid \underset{\vee}{10}, \overset{\wedge}{13}, 9, 7 \mid \underset{\vee}{2}, 4, \overset{\wedge}{19}, 5, 3,$$
$$0 \qquad 1 \qquad 2 \qquad 3 \qquad 4$$

already trough-factorized, step (1) yields

$$1, 6, 11, \overset{\wedge}{18}, 17 \mid \overset{\wedge}{20}, 15, \underset{\vee}{16} \mid \overset{\wedge}{14}, \underset{\vee}{8}, 12 \mid \overset{\wedge}{13}, 9, 7, \underset{\vee}{10} \mid \overset{\wedge}{19}, 5, 3, \underset{\vee}{2}, 4.$$
$$0 \qquad 1 \qquad 2 \qquad 3 \qquad 4$$

In step (2) we define: $\psi^-(17) = 0$, $\psi^-(15) = 0$, $\psi^-(9) = 2$, $\psi^-(7) = 0$, $\psi^-(5) = 4$, $\psi^-(3) = 4$. In step (3) we get: $\overline{d}_0 = 17, 15, 7$; $\overline{d}_1 = e$; $\overline{d}_2 = 9$; $\overline{d}_3 = e$; $\overline{d}_4 = 5, 3$. As $\overline{d}_0 = 17, 15, 7$ is nonempty, we apply (4″) with $h_0 = 1, 6, 11$ and $\overline{d'_0} = 7$. We get $h'_0 = 1, 6 < 7 < 11 = h''_0$. Hence

$$\mathbf{D}\,\sigma = (\infty, 0, 1, 6)(18, 17, 15, 7, 11)(20, 16)(14, 9, 8, 12)(13, 10)(19, 5, 3, 2, 4).$$

Several properties are to be verified.

(a) *Each letter in* $\mathbf{D}\,\sigma$ *is a oiseau*. First, $0 \ll h_0 \ll p_0$, so $(\infty 0 h_0 p_0)$ is a oiseau. Next, for each $i = 1, 2, \ldots, r$ we have $p_i \gg \overline{d}_i \gg t_i$ by (2)–(3). Also $t_i \ll h_i \ll p_i$, since t_i is a trough and p_i a peak in the permutation σ. Consequently, for each $i = 1, 2, \ldots, r$, each letter $(p_i\overline{d}_it_ih_i)$ is a oiseau.

Now, if x is a letter in d_0, it is also a letter in $\overline{d}_0$ by (2), so $p_0 \gg x \gg t_1$. Furthermore, if x is a letter of some d_i with $1 \le i \le r$ and $\psi^-(x) = 0$, then $x < t_1, x < t_2, \ldots, x < t_i$.

As $p_0 \gg t_1$, we conclude that $p_0 \gg x$. Hence, $p_0 \gg \overline{d}_0$ and $(p_0\overline{d}_0 h_0'') = (p_0\overline{d_0'}t_0 h_0'')$ is also a oiseau.

(b) *The word* $\mathbf{D}\,\sigma$ *is* C*-maximal*. First $(\infty 0 h_0 p_0)$ (resp. $(\infty 0 h_0')$) does not commute with any one of the other letters in $\mathbf{D}\,\sigma$. As t_{i+1} is the closest trough to the right of the peak p_i in σ, we have $p_i > t_{i+1} (i \ge 1)$. Hence $p_i\overline{d}_i t_i h_i \ll p_{i+1}\overline{d}_{i+1}t_{i+1}h_{i+1}$ cannot hold. It then follows that if $(p_i\overline{d}_i t_i h_i)$ and $(p_{i+1}\overline{d}_{i+1}t_{i+1}h_{i+1})$ commute, we necessarily have $p_i\overline{d}_i t_i h_i \gg p_{i+1}\overline{d}_{i+1}t_{i+1}h_{i+1}$. Finally, as $p_0 > t_1$, we show, using the same argument, that if $(p_0\overline{d_0'}t_0 h_0'')$ and $(p_1\overline{d}_1 t_1 h_1)$ commute, we have $(p_0\overline{d_0'}t_0 h_0'') \gg (p_1\overline{d}_1 t_1 h_1)$. Thus $\mathbf{D}\,\sigma$ is C-maximal.

(c) *Construction of the inverse* $\mathbf{D}^{-1}$. Compare steps (4′) and (4″). In (4′) the rightmost letter p_0 in the first oiseau is greater than the trough t_1 in the second oiseau. In (4″) the rightmost letter of $0h_0'$ in the first oiseau is less than the trough t_0 in the second oiseau. This makes up the first step in the definition of the reverse of $\mathbf{D}$. Let $v = \omega_0\omega_1\ldots\omega_s$ belong to $B_n^{\max}$ with $\omega_0 = (\infty 0 k_0)$. If $s = 0$, then $\mathbf{D}^{-1}\,v = k_0 = 1, 2, \ldots, n$. Assume now that $s \ge 1$.

(i) If the rightmost letter of $0k_0$ is *not* less than the trough in the oiseau ω_1, it cannot be 0; denote it by p_0. We may express v as

$$v = (\infty 0 h_0 p_0)(p_1\overline{d}_1 t_1 h_1)(p_2\overline{d}_2 t_2 h_2)\ldots(p_r\overline{d}_r t_r h_r), \tag{5.1}$$

with $r = s$, $0h_0p_0 = 0k_0$ and $(p_i\overline{d}_i t_i h_i) = \omega_i$ for $i = 1, 2, \ldots, r$.

(ii) If the rightmost letter of $0k_0$ is less than the trough in the oiseau ω_1, we may express v as

$$v = (\infty 0 h_0')(p_0\overline{d_0'}t_0 h_0'')(p_1\overline{d}_1 t_1 h_1)(p_2\overline{d}_2 t_2 h_2)\ldots(p_r\overline{d}_r t_r h_r), \tag{5.2}$$

with $r = s - 1$, $\omega_0 = (\infty 0 k_0) = (\infty 0 h_0')$, $\omega_1 = (p_0\overline{d_0'}t_0 h_0'')$ and $\omega_{i+1} = (p_i\overline{d}_i t_i h_i)$ for $i = 0, 1, \ldots, r$ and $0h_0' \ll t_0$. Then define $h_0 := h_0' h_0''$, $\overline{d}_0 := \overline{d_0'}t_0$.

In both cases the words $\overline{d}_i$, h_i and the letter p_i have been defined for each $i = 0, 1, \ldots, r$ and the letter t_i for $i = 1, 2\ldots, r$. We form the factorization

$$h_0 p_0 \overline{d}_0 \mid t_1 h_1 p_1 \overline{d}_1 \mid t_2 h_2 p_2 \overline{d}_2 \mid \cdots \mid t_r h_r p_r \overline{d}_r. \tag{5.3}$$

With the convention $t_{r+1} = -\infty$ and for $i = 0, 1, \ldots, r$ we move each letter x of $\overline{d}_i$ *to the right* until it falls between a peak p_j and the next trough t_{j+1}. More formally, we define $\psi(x)$ to be the least integer j such that $i \le j \le r$ and $p_j > x > t_{j+1}$ and let d_j be the *decreasing* word with all the letters x such that $\psi(x) = j$. As $t_i \ll h_i \ll p_i$, because the letters of v are oiseaux, the factorization

$$h_0 p_0 d_0 \mid t_1 h_1 p_1 d_1 \mid t_2 h_2 p_2 d_2 \mid \cdots \mid t_r h_r p_r d_r \tag{5.4}$$

is a trough factorization of a permutation. This defines $\mathbf{D}^{-1}\,v$. We also convince ourselves that $\mathbf{D}^{-1}\mathbf{D}$ and $\mathbf{D}\mathbf{D}^{-1}$ are identity maps.

Example 1. Let

$$v = (\infty, 0, 18, 20)(17, \mathbf{15, 6}, 3, 7)(19, 16)(14, \mathbf{10, 8}, 2, 12)(13, \mathbf{11}, \mathbf{9})(\mathbf{5}, \mathbf{4}, \mathbf{1})$$

belong to $B_{20}^{\max}$. As the rightmost letter 20 of the factor 0, 18, 20 is not less than the trough 3 of the second oiseau, step (i) applies. We define $h_0 = 18, 20$, $p_0 = 20$, $\overline{d}_1 = \mathbf{15, 6}$, $\overline{d}_2 = e$,

$\overline{d}_3 = \mathbf{10, 8}$, $\overline{d}_4 = \mathbf{11}$, $\overline{d}_5 = \mathbf{4}$. Next, form the factorization introduced in (5.3):

$$18, 20 \mid 3, 7, 17, \mathbf{15, 6} \mid 16, 19 \mid 2, 12, 14, \mathbf{10, 8} \mid 9, 13, \mathbf{11} \mid 1, 5, \mathbf{4}$$

and move the letters of the $\overline{d}_i$'s (the letters in bold-face) to the right until they are inserted correctly between a peak and the next trough. We get

$$18, 20 \mid 3, 7, 17 \mid 16, 19, \mathbf{15, 6} \mid 2, 12, 14, \mathbf{10} \mid 9, 13, \mathbf{11, 8} \mid 1, 5, \mathbf{4}.$$

This is the trough factorization of the permutation $\mathbf{D}^{-1}v$.

Example 2. The following word:

$$v = (\infty, 0, 1, 6)(18, 17, 15, 7, 11)(20, 16)(14, 9, 8, 12)(13, 10)(19, 5, 3, 2, 4)$$

belongs to $B_{20}^{\max}$. The rightmost letter 6 of the factor 0, 1, 6 is less than the trough 7 of the second oiseau, so step (ii) applies. We have $h_0' = 1, 6$, $p_0 = 18$, $\overline{d_0'} = 17, 15$, $t_0 = 7$, $h_0'' = 11$; then $h_0 = h_0' h_0'' = 1, 6, 11$, $\overline{d}_0 = \overline{d_0'} t_0 = \mathbf{17, 15, 7}$, $\overline{d}_1 = e$, $\overline{d}_2 = \mathbf{9}$, $\overline{d}_3 = e$, $\overline{d}_4 = \mathbf{5, 3}$. Thus, the (5.1)-factorization reads

$$1, 6, 11, 18, \mathbf{17, 15, 7} \mid 16, 20 \mid 8, 12, 14, \mathbf{9} \mid 10, 13 \mid 2, 4, 19, \mathbf{5, 3}.$$

Next, we move the letters in bold-face to the right, following the rule described before. We get

$$1, 6, 11, 18, \mathbf{17} \mid 16, 20, \mathbf{15} \mid 8, 12, 14 \mid 10, 13, \mathbf{9, 7} \mid 2, 4, 19, \mathbf{5, 3},$$

which is the trough factorization of a permutation $\sigma = \mathbf{D}^{-1}v$.

6. Order statistics

Let $0\sigma = 0x_1x_2 \dots x_n$ be a permutation, starting with 0, whose peak factorization reads

$$t_0h_0 \mid p_1d_1t_1h_1 \mid p_2d_2t_2h_2 \mid \dots \mid p_rd_rt_rh_r.$$

Let $p_0 := \infty$, $d_0 := e$, $t_0 = 0$ and for $0 \le i < j \le r$ let $|p_id_it_ih_i|_{]t_j, p_j[}$ denote the number of letters in the word $p_id_it_ih_i$ that belong to the open interval $]t_j, p_j[$. As only the factors $p_id_it_i$ contain subfactors of the form ca, we can express $(b - ca)\sigma$ as

$$(b - ca)\sigma = \sum_{1\le j\le r} \sum_{i<j} |p_id_it_ih_i|_{]t_j, p_j[}. \tag{6.1}$$

Now, let $v = (\infty 0 h_0)(p_1d_1t_1h_1') \dots (p_rd_rt_rh_r')$ belong to the C-equivalence class B_n and define

$$(b - ca)v := \sum_{1\le j\le r} \sum_{i<j} |p_id_it_ih_i'|_{]t_j, p_j[}, \tag{6.2}$$

keeping the convention $p_0 := \infty$, $d_0 := e$, $t_0 = 0$. When two oiseaux $(p_id_it_ih_i')$, $(p_jd_jt_jh_j')$ commute, $|p_id_it_ih_i'|_{]t_j, p_j[} = 0$. It then follows that if two elements v, v' in A^*, of valuation 1^n, are C-equivalent, then

$$(b - ca)v = (b - ca)v'. \tag{6.3}$$

When we go from σ to $\mathbf{d}\,\sigma$, only the *positions* of some double-rises y change, and if they do, those double-rises move *to the left*, but not to the left of factors $p_i d_i t_i h_i$ such that $p_i > y$, as shown in (4.2). Hence, we also have

$$|p_i d_i t_i h_i'|_{]t_j, p_j[} = |p_i d_i t_i h_i|_{]t_j, p_j[}$$

for $0 \le i < j \le r$. Hence,

$$(b - ca)\sigma = (b - ca)\mathbf{d}\,\sigma. \tag{6.4}$$

But as the statistic $(b - ca)$ depends only on the C-equivalence class and not on its representative by (6.3), we also have

$$(b - ca)\theta(\mathbf{d}\,\sigma) = (b - ca)\mathbf{d}\,\sigma. \tag{6.5}$$

The next proposition is the analog for $\mathbf{D}$ of property (6.4). It requires a longer proof.

Proposition 6.1. *Let σ be a permutation of order n. Then*

$$(b - ca)\sigma - (b - ca)\mathbf{D}\,\sigma = n - \mathrm{L}\,\sigma - \mathrm{des}\,\sigma. \tag{6.6}$$

Proof. Let $v := \mathbf{D}\,\sigma$. As seen in Section 5 (c), the element v is in $B_n^{\max}$ and appears in the form (5.1) or in the form (5.2). When $r = 0$, we have $v = (\infty 0 h_0 p_0)$ or $v = (\infty 0 h_0')(p_0 \overline{d_0'} t_0 h_0'')$. In the first case, $\sigma = h_0 p_0$ is the increasing word $12\ldots n$ and identity (6.6) holds. In the second case, $\sigma = h_0' h_0'' p_0 \overline{d_0'} t_0$ and $\mathrm{des}\,\sigma = |p_0 \overline{d_0'}|$, $(b - ca)\sigma = |h_0' h_0''|_{]t_0, p_0[} = |h_0''|_{]t_0, p_0[} = |h_0''|$. On the other hand, $(b - ca)\,\mathbf{D}\,\sigma = |h_0'|_{]t_0, p_0[} = 0$, since $h_0' \ll t_0$, whenever h_0' is nonempty. Thus, the left-hand side of (6.6) is equal to $|h_0''|$.

To evaluate the right-hand side of (6.6) consider two cases: (i) $h_0' = h_0'' = e$; (ii) h_0', h_0'' not both empty. In case (i) $n - \mathrm{L}\,\sigma - \mathrm{des}\,\sigma = p_0 - 1 - (p_0 - 1) = 0 = |h_0''|$, since $p_0 \overline{d_0'} t_0 = n(n-1)\ldots 1$. In case (ii) we have $n - \mathrm{L}\,\sigma - \mathrm{des}\,\sigma = p_0 - 1 - |p_0 \overline{d_0'}| = p_0 - |p_0 \overline{d_0'} t_0|$, which is equal to the number of letters in σ less than t_0. This number is precisely $|h_0''|$. Thus (6.6) holds for $r = 0$.

We then proceed by induction on r and assume that $r \ge 1$. The construction of $\mathbf{D}$, given in Section 5, depends only on the total ordering on the set $\{1, 2, \ldots, n\}$. The latter set can then be replaced by any set E of integers of the same cardinality. Let $\mathcal{S}_E$ be the set of the permutations of E. Then the set $\mathbf{D}(\mathcal{S}_E)$ will again be denoted by $B_n^{\max}$, the context indicating the underlying set E that is in use.

Having this convention in mind we can write v, using the notation (5.1) and (5.2), as $v = v_1(p_r \overline{d}_r t_r h_r)$, where v_1 is an element of $B_{n_1}^{\max}$ for some $n_1 \le n - 2$. Let $\sigma_1 := \mathbf{D}^{-1} v_1$. Again notice that σ_1 is a permutation of a set of cardinality n_1. We shall evaluate the left-hand side of (6.6) as the sum $U_1 + U_2 - U_3$, where

$$U_1 := (b - ca)\sigma - (b - ca)\sigma_1; \qquad U_2 := (b - ca)\sigma_1 - (b - ca)v_1;$$
$$U_3 := (b - ca)v - (b - ca)v_1.$$

A first evaluation of U_3. Using (5.1) and (5.2) let

$$w := h_0 p_0 p_1 \overline{d}_1 t_1 h_1 \ldots p_{r-1} \overline{d}_{r-1} t_{r-1} h_{r-1}$$

(resp. $h'_0 p_0 \overline{d'_0} t_0 h''_0 p_1 \overline{d}_1 t_1 h_1 \dots p_{r-1} \overline{d}_{r-1} t_{r-1} h_{r-1}$). Then

$$(b - ca)v - (b - ca)v_1 = |w|_{]t_r, p_r[}. \tag{6.7}$$

Evaluation of U_1. By (5.4), $\sigma = h_0 p_0 d_0 t_1 h_1 p_1 d_1 \dots t_r h_r p_r d_r$. In the construction of $\mathbf{D}^{-1}\sigma$ the decreasing factor d_r is made of the double-descents x in w such that $\psi(x) = r$ (let d'_{r-1} be the decreasing word made of those letters x), plus all the double-descents in the factor $\overline{d}_r$. Also, by construction,

$$d'_{r-1} \ll t_r \tag{6.8}$$

and also,

$$t_r \ll \overline{d}_r, \tag{6.9}$$

since $(p_r d_r t_r \overline{h}_r)$ is a oiseau. Hence

$$d_r = \overline{d}_r d'_{r-1}. \tag{6.10}$$

Accordingly,

$$\sigma = h_0 p_0 d_0 t_1 h_1 p_1 d_1 \dots t_{r-1} h_{r-1} p_{r-1} d_{r-1} t_r h_r p_r \overline{d}_r d'_{r-1};$$
$$\sigma_1 = h_0 p_0 d_0 t_1 h_1 p_1 d_1 \dots t_{r-1} h_{r-1} p_{r-1} d_{r-1} d'_{r-1}.$$

We can evaluate U_1, the open intervals to be considered being $]t_r, p_{r-1}[$ and $]\mathrm{L}\,\sigma, p_r[$ for σ and $]\mathrm{L}\,\sigma_1, p_{r-1}[$ for σ_1. We get

$$\begin{aligned} U_1 = {} & |\sigma_1|_{]t_r, p_{r-1}[} - |p_{r-1} d_{r-1} d'_{r-1}|_{]t_r, p_{r-1}[} \\ & + |\sigma_1|_{]\mathrm{L}\,\sigma, p_r[} - |d'_{r-1}|_{]\mathrm{L}\,\sigma, p_r[} + |t_r h_r|_{]\mathrm{L}\,\sigma, p_r[} \\ & - |\sigma_1|_{]\mathrm{L}\,\sigma_1, p_{r-1}[} + |p_{r-1} d_{r-1} d'_{r-1}|_{]\mathrm{L}\,\sigma_1, p_{r-1}[}. \end{aligned} \tag{6.11}$$

Evaluation of U_2. As the bottoms of the descents are the t_i's and the double-descents, we have $\operatorname{des}\sigma - \operatorname{des}\sigma_1 = |d_{r-1} t_r| + |\overline{d}_r d'_{r-1}| - |d_{r-1} d'_{r-1}| = |t_r \overline{d}_r|$. On the other hand, $n - \mathrm{L}\,\sigma = |\sigma|_{]\mathrm{L}\,\sigma, +\infty[}$, so by induction

$$\begin{aligned} U_2 &= |\sigma_1|_{]\mathrm{L}\,\sigma_1, +\infty[} - \operatorname{des}\sigma_1 \\ &= |\sigma_1|_{]\mathrm{L}\,\sigma_1, +\infty[} - \operatorname{des}\sigma + |t_r \overline{d}_r|. \end{aligned} \tag{6.12}$$

Evaluation of U_3. Again take the expression derived in (6.7). The word w is a rearrangement of $h_0 p_0 \overline{d}_0 t_1 h_1 p_1 \overline{d}_1 \dots t_{r-1} h_{r-1} p_{r-1} \overline{d}_{r-1}$ [see (5.3)] and also of $h_0 p_0 d_0 t_1 h_1 p_1 \dots t_{r-1} h_{r-1} p_{r-1} d_{r-1} d'_{r-1} = \sigma_1$ [see (5.4)] by definition of d'_{r-1}. Hence

$$U_3 = |\sigma_1|_{]t_r, p_r[}. \tag{6.13}$$

To calculate $U_1 + U_2 - U_3$, two cases are to be considered: (i) $\mathrm{L}\,\sigma = \mathrm{L}\,\sigma_1$; (ii) $\mathrm{L}\,\sigma \neq \mathrm{L}\,\sigma_1$.

Case (i): The factor d'_{r-1} is nonempty. By (6.8) we have $\mathrm{L}\,\sigma = \mathrm{L}\,d'_{r-1} < t_r$ and

$$-|p_{r-1} d_{r-1} d'_{r-1}|_{]t_r, p_{r-1}[} - |d'_{r-1}|_{]\mathrm{L}\,\sigma, p_r[} + |p_{r-1} d_{r-1} d'_{r-1}|_{]\mathrm{L}\,\sigma, p_{r-1}[} = 0.$$

There are three terms of the form $|\sigma|_{]\dots[}$ in (6.11) and one in (6.13) and their contribution to $U_1 - U_3$ is

$$|\sigma_1|_{]t_r, p_{r-1}[} + |\sigma_1|_{]\mathrm{L}\,\sigma, p_r[} - |\sigma_1|_{]\mathrm{L}\,\sigma_1, p_{r-1}[} - |\sigma_1|_{]t_r, p_r[} = 0,$$

since $\mathrm{L}\,\sigma = \mathrm{L}\,\sigma_1$. Hence

$$\begin{aligned} U_1 + U_2 - U_3 &= |t_r h_r|_{]\mathrm{L}\,\sigma, p_r[} + |\sigma_1|_{]\mathrm{L}\,\sigma,+\infty[} - \operatorname{des}\sigma + |t_r \overline{d}_r| \\ &= |\sigma_1|_{]\mathrm{L}\,\sigma,+\infty[} + |p_r \overline{d}_r t_r h_r| - \operatorname{des}\sigma \\ &= |\sigma|_{]\mathrm{L}\,\sigma,+\infty[} - \operatorname{des}\sigma. \end{aligned}$$

Case (ii): The factor d'_{r-1} is then empty and $\mathrm{L}\,\sigma_1 = \mathrm{L}\ p_{r-1}d_{r-1} > t_r$, $\mathrm{L}\,\sigma > t_r$, $\overline{d}_r = d_r$. Then

$$\begin{aligned} &|\sigma_1|_{]\mathrm{L}\,\sigma_1,+\infty[} + |\sigma_1|_{]t_r, p_{r-1}[} + |\sigma_1|_{]\mathrm{L}\,\sigma, p_r[} - |\sigma_1|_{]\mathrm{L}\,\sigma_1, p_{r-1}[} - |\sigma_1|_{]t_r, p_r[} \\ &\quad = |\sigma_1|_{]\mathrm{L}\,\sigma_1,+\infty[} + |\sigma_1|_{]t_r,\mathrm{L}\,\sigma_1]} - |\sigma_1|_{]t_r,\mathrm{L}\,\sigma]} \\ &\quad = \begin{cases} |\sigma_1|_{]\mathrm{L}\,\sigma_1,+\infty[} + |\sigma_1|_{]\mathrm{L}\,\sigma,\mathrm{L}\,\sigma_1]}, & \text{if} \quad \mathrm{L}\,\sigma < \mathrm{L}\,\sigma_1; \\ |\sigma_1|_{]\mathrm{L}\,\sigma_1,+\infty[} - |\sigma_1|_{]\mathrm{L}\,\sigma_1,\mathrm{L}\,\sigma]}, & \text{if} \quad \mathrm{L}\,\sigma_1 < \mathrm{L}\,\sigma. \end{cases} \\ &\quad = |\sigma_1|_{]\mathrm{L}\,\sigma,+\infty[}. \end{aligned}$$

Also

$$-|p_{r-1}d_{r-1}d'_{r-1}|_{]t_r, p_{r-1}[} - |d'_{r-1}|_{]\mathrm{L}\,\sigma, p_r[} + |p_{r-1}d_{r-1}d'_{r-1}|_{]\mathrm{L}\,\sigma_1, p_{r-1}[} = -1.$$

Hence

$$\begin{aligned} U_1 + U_2 - U_3 &= |\sigma_1|_{]\mathrm{L}\,\sigma,+\infty[} - \operatorname{des}\sigma + |t_r h_r|_{]\mathrm{L}\,\sigma, p_r[} + |t_r \overline{d}_r| - 1 \\ &= |\sigma_1|_{]\mathrm{L}\,\sigma,+\infty[} - \operatorname{des}\sigma + |p_r d_r t_r h_r|_{]\mathrm{L}\,\sigma,+\infty[} \\ &= |\sigma|_{]\mathrm{L}\,\sigma,+\infty[} - \operatorname{des}\sigma. \quad \square \end{aligned}$$

With the completion of Proposition 6.1, all the statements of Theorem 1.1 have been proved, except statement (iv). In fact, only the steps $(4')$ and $(4'')$ in the definition of $\mathbf{D}$ (see Section 5) modify the nature of some peaks, troughs, double-descents or double-rises. In $(4')$ the peak p_0 becomes a double-rise, but troughs and double-descents remain alike. In $(4'')$ the rightmost letter of $\overline{d}_0$, a descent, becomes a trough. In both cases, the *set* of troughs and double-descents is not modified. This proves statement (iv).

7. Euler–Mahonian statistics

Let us go back to the notation introduced by Babson and Steingrímsson [1] for "atomic" permutation statistics. For each permutation α, β, γ of a, b, c the expression $(\alpha - \beta\gamma)(w)$ is meant to be the number of pairs (i, j), $1 \le i < j < n$, such that the orderings of the two triples (x_i, x_j, x_{j+1}) and (α, β, γ) are identical. Of course, $(ba)\sigma$ denotes the number of *descents*, $\operatorname{des}\sigma$ (i.e. the number of places $1 \le i < n$ such that $x_i > x_{i+1}$). Finally, $[b-a)\sigma$ is the number of x_j such that $x_1 > x_j$, that is, $x_1 - 1$.

Using these atomic objects, Babson and Steingrímsson noticed that the two classical *Mahonian* statistics, the inversion number "inv" and the major index "maj", could be written in terms of those patterns. For instance, $\mathrm{inv} = (bc-a)+(ca-b)+(cb-a)+(ba)$ and $\mathrm{maj} = (a - cb) + (b - ca) + (c - ba) + (ba)$. It is worth mentioning that an analogous notation has been introduced by Han ([8, Theorem 2.1]) that enabled him to make a link between the major index and the Denert statistic. The former authors were then motivated to perform a computer search for all statistics that could be thus written and be "conjecturally Mahonian". Several of the statistics they have introduced have been shown to be Mahonian since. However, their Conjecture 11 has remained open (it will be stated later on). Our purpose will be to prove its correctness by using the previous algebraic set-up. Before doing so let us recall some basic notation and facts for Euler–Mahonian statistics.

As usual,

$$(a;q)_n := \begin{cases} 1, & \text{if} \quad n = 0; \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & \text{if} \quad n \geq 1; \end{cases} \tag{7.1}$$

$$(a;q)_\infty := \lim_n (a;q)_n = \prod_{n\geq 0}(1-aq^n). \tag{7.2}$$

Also,

$$[n]_q := \frac{1-q^n}{1-q} = (1+q+\cdots+q^{n-1}); \tag{7.3}$$

$$\begin{aligned} [n]_q! &:= \frac{(q;q)_n}{(1-q)^n} = [n]_q[n-1]_q\cdots[1]_q \\ &= (1+q+\cdots+q^{n-1})(1+q+\cdots+q^{n-2})\cdots(1). \end{aligned} \tag{7.4}$$

The sequence of polynomials $([n]_q!)$ $(n \geq 0)$ is said to be *Mahonian*. On the other hand, a statistic "stat" (actually, we should say a sequence (stat_n) $(n \geq 0)$ of statistics, where stat_n is defined on the symmetric group $\mathcal{S}_n$) is said to be *Mahonian* if for every $n \geq 0$ we have

$$\sum_{w\in\mathcal{S}_n} q^{\mathrm{stat}\,w} = [n]_q!.$$

A sequence $(A_n(t,q))(n \geq 0)$ of polynomials in two variables, t and q, is said to be *Euler–Mahonian*, if one of the following *equivalent* conditions holds:

(1) For every $n \geq 0$,

$$\frac{1}{(t;q)_{n+1}} A_n(t,q) = \sum_{s\geq 0} t^s([s+1]_q)^n. \tag{7.5}$$

(2) The exponential generating function for the fractions $A_n(t,q)/(t;q)_{n+1}$ is given by

$$\sum_{n\geq 0}\frac{u^n}{n!}\frac{A_n(t,q)}{(t;q)_{n+1}} = \sum_{s\geq 0} t^s \exp(u[s+1]_q). \tag{7.6}$$

(3) The sequence $(A_n(t,q))$ satisfies the recurrence relation

$$(1-q)A_n(t,q) = (1-tq^n)A_{n-1}(t,q) - q(1-t)A_{n-1}(tq,q). \tag{7.7}$$

(4) Let $A_n(t,q) = \sum_{s\geq 0} t^s A_{n,s}(q)$. Then the coefficients $A_{n,s}(q)$ satisfy the recurrence

$$A_{n,s}(q) = [s+1]_q A_{n-1,s}(q) + q^s[n-s]_q A_{n-1,s-1}(q). \tag{7.8}$$

It is routine to prove that those four conditions are equivalent (see, e.g., [5, Sections 6, 7] for a proof in a more general setting). Now a pair of statistics $(\mathrm{stat}_1, \mathrm{stat}_2)$ defined on each symmetric group $\mathcal{S}_n$ ($n \geq 0$) is said to be *Euler–Mahonian* if for every $n \geq 0$ we have

$$\sum_{w\in\mathcal{S}_n} t^{\mathrm{stat}_1 w} q^{\mathrm{stat}_2 w} = A_n(t,q).$$

The first values of the Euler–Mahonian polynomials are the following:

$$\begin{aligned}
&A_0(t,q) = A_1(t,q) = 1; \quad A_2(t,q) = 1 + tq;\\
&A_3(t,q) = 1 + t(2q + 2q^2) + t^2q^3;\\
&A_4(t,q) = 1 + t(3q + 5q^2 + 3q^3) + t^2(3q^3 + 5q^4 + 3q^5) + t^3q^6;\\
&A_5(t,q) = 1 + t(4q + 9q^2 + 9q^3 + 4q^4) + t^2(6q^3 + 16q^4 + 22q^5 + 16q^6 + 6q^7)\\
&\qquad\qquad + t^3(4q^6 + 9q^7 + 9q^8 + 4q^9) + t^4q^{10}.
\end{aligned}$$

The bistatistic that is Euler–Mahonian *par excellence* is the pair $(\mathrm{des}\,,\mathrm{maj})$. The proof of this statement goes back to Carlitz [2,3], although some q-calculations for the major index already appeared in [9]. Further bistatistics have since been introduced and proved to be Euler–Mahonian, in particular the pair $(\mathrm{des}\,,\mathrm{mak})$ (see [7]), where

$$\mathrm{mak} := (a - cb) + (b - ca) + (cb - a) + (ba). \tag{7.9}$$

Actually, the original "mak", as introduced by Foata and Zeilberger [7], reads

$$(a - cb) + (ca - b) + (cb - a) + (ba),$$

but Babson and Steingrímsson changed the notation. However, we shall stick with Definition (7.9) in the following. Notice that to prove that (des, mak) is Euler–Mahonian requires only a slight modification in the argument used in [7, Section 8]. Babson–Steingrímsson's Conjecture 11 is the following.

Conjecture 11. *Let*

$$\begin{aligned}
S_1 &:= (a - cb) + (b - ac) + (cb - a) + [b - a),\\
S_2 &:= (a - cb) + (b - ac) + (c - ba) + [b - a),\\
S_3 &:= (a - cb) + (b - ca) + (cb - a) + [b - a),\\
S_4 &:= (a - cb) + (b - ca) + (c - ba) + [b - a).
\end{aligned}$$

Then the bistatistics $(\mathrm{des}\,, S_i)$ *for* $i = 1, 2, 3, 4$ *are Euler–Mahonian.*

In the next section we prove that $(\mathrm{des}\,, S_1)$ and $(\mathrm{des}\,, S_3)$ are equidistributed with $(\mathrm{des}\,,\mathrm{mak})$ (this requires the machinery developed in the previous sections). The last section is devoted to proving that $(\mathrm{des}\,, S_2)$ and $(\mathrm{des}\,, S_4)$ are equidistributed with $(\mathrm{des}\,,\mathrm{maj})$. This is far easier.

8. The transformation Φ

The transformation $\Phi := \mathbf{D}^{-1}\theta\,\mathbf{d}$, already introduced in the introduction, maps the permutation group $\mathcal{S}_n$ onto itself. Let Φ^{-1} be the inverse bijection. As is often the case, such combinatorial bijections are now combined with the classical dihedral transformations, namely the *reverse image* $\mathbf{r}$ and the *complement* $\mathbf{c}$ that map each permutation $\sigma = x_1x_2\dots x_n$ onto $\mathbf{r}\,\sigma = x_n\dots x_2x_1$ and $\mathbf{c}\,\sigma = (n+1-x_1)(n+1-x_2)\dots(n+1-x_n)$, respectively.

Denote by $\mathrm{Dbv}\,\sigma$ the sum of the *descent bottom values* of the permutation σ, that is, the sum of the troughs and double-descents of σ. The following formula can be obtained by means of a simple counting:

$$\mathrm{Dbv} = (a-cb)+(cb-a)+(ba). \tag{8.1}$$

It follows from (7.9) that "mak" can also be expressed as

$$\mathrm{mak} = \mathrm{Dbv}+(b-ca). \tag{8.2}$$

Another statistic S will be used for technical reasons. It is defined by

$$S := \mathrm{Dbv}+(b-ca)+(ba)+L-n = \mathrm{mak}+(ba)+L-n. \tag{8.3}$$

Statement (iv) of Theorem 1.1, that was proved at the end of Section 6, has two consequences, essential for the subsequent derivation:

$$\mathrm{des}\,\Phi(\sigma) = \mathrm{des}\,\sigma \qquad (\text{or } (ba)\,\Phi(\sigma) = (ba)\sigma); \tag{8.4}$$

$$\mathrm{Dbv}\,\Phi(\sigma) = \mathrm{Dbv}\sigma. \tag{8.5}$$

Theorem 8.1. *The transformation $\mathbf{r}\,\Phi^{-1}\,\mathbf{r}\,\Phi$ is a bijection of $\mathcal{S}_n$ onto itself having the property that*

$$(\mathrm{des}\,,S_1)(\mathbf{r}\,\Phi^{-1}\,\mathbf{r}\,\Phi)(\sigma) = (\mathrm{des}\,,\mathrm{mak})\,\sigma \tag{8.6}$$

holds for each permutation σ.

Theorem 8.2. *The transformation $\mathbf{r}\,\mathbf{c}\,\Phi\,\mathbf{r}\,\mathbf{c}$ is a bijection of $\mathcal{S}_n$ onto itself having the property that*

$$(\mathrm{des}\,,S_3)(\mathbf{r}\,\mathbf{c}\,\Phi\,\mathbf{r}\,\mathbf{c})(\sigma) = (\mathrm{des}\,,\mathrm{mak})\,\sigma \tag{8.7}$$

holds for each permutation σ.

The first step in the proof of Theorem 8.1 is to establish the following identity:

$$\mathrm{mak}\,\mathbf{r}\,\Phi(\sigma) - \mathrm{mak}\,\mathbf{r}\,\sigma = \mathrm{L}\,\sigma + \mathrm{F}\,\sigma - \mathrm{F}\,\Phi(\sigma) + \mathrm{des}\,\Phi(\sigma) - n, \tag{8.8}$$

for each $\sigma \in \mathcal{S}_n$.

By definition of "mak" we get

$$\mathrm{mak}\,\mathbf{r} = (bc-a)+(ac-b)+(a-bc)+(ab).$$

Let $T_1 := (a-bc)+(bc-a)+(ab)$ and $T_2 := (b-ac)+(ac-b)$, so that $\mathrm{mak}\,\mathbf{r} = T_1+T_2-(b-ac)$. Let us evaluate $T_1\,\sigma$, $T_2\,\sigma$ in terms of the factors of the *peak factorization*

$$0h_0 \mid p_1d_1t_1h_1 \mid p_2d_2t_2h_2 \mid \dots \mid p_rd_rt_rh_r$$

of 0σ introduced in (4.1). When $\sigma'' = \Phi(\sigma)$, let us also calculate $T_1\,\sigma''$, $T_2\,\sigma''$ in terms of the *trough factorization*

$$h_0''p_0''d_0'' \mid t_1''h_1''p_1''d_1'' \mid t_2''h_2''p_2''d_2'' \mid \dots \mid t_s''h_s''p_s''d_s'',$$

of σ'' introduced in Section 5. We have added a double prime to all the factors to avoid any confusion with the factorization of σ. As $\theta(\sigma)$ is a rearrangement of the oiseau-word $\mathbf{d}\,\sigma$ and $\sigma'' = \mathbf{D}^{-1}\theta\,\mathbf{d}\,\sigma$, we have

$$\theta(\mathbf{d}\,\sigma) = \mathbf{D}\,\sigma'' = (\infty 0 h'_0)(p_{i_1} d_{i_1} t_{i_1} h'_{i_1}) \ldots (p_{i_r} d_{i_r} t_{i_r} h'_{i_r}), \tag{8.9}$$

where, as seen in (4.3), the word $h'_0 h'_1 h'_2 \ldots h'_r$ (and also $h'_0 h'_{i_1} h'_{i_2} \ldots h'_{i_r}$) is a rearrangement of $h_0 h_1 h_2 \ldots h_r$.

If $x_i < x_{i+1}$ holds in the permutation $\sigma = x_1 x_2 \ldots x_n$, then x_i is either a trough or a double-rise. The contribution of x_i to $T_1\,\sigma$ is equal to x_i if $i \le n-1$ and 0 if $i = n$. As, by convention, x_n is either a trough or a double-rise, we get

$$T_1\,\sigma = \sum_{i=0}^{r} \operatorname{tot} h_i + \sum_{i=1}^{r} t_i - \mathrm{L}\,\sigma.$$

The same investigation of the contribution of each factor $h_i\, p_{i+1}$ to $T_2\,\sigma$ leads to the evaluation

$$\begin{aligned} T_2\,\sigma &= (p_1 - \mathrm{F}\,\sigma - |h_0|) + \sum_{i=2}^{r} (p_i - t_{i-1} - |h_{i-1}| - 1) + (\mathrm{L}\,\sigma - t_r - |h_r|) \\ &= \sum_{i=1}^{r} p_i - \sum_{i=1}^{r} t_i - \sum_{i=0}^{r} |h_i| + \mathrm{L}\,\sigma - \mathrm{F}\,\sigma - (r-1), \end{aligned}$$

so we have

$$(T_1 + T_2)\sigma = \sum_{i=1}^{r} p_i + \sum_{i=0}^{r} \operatorname{tot} h_i - \sum_{i=0}^{r} |h_i| - \mathrm{F}\,\sigma - r + 1. \tag{8.10}$$

In the same manner,

$$\begin{aligned} T_1\,\sigma'' &= \sum_{i=0}^{s} \operatorname{tot} h''_i + \sum_{i=1}^{s} t''_i; \\ T_2\,\sigma'' &= (p''_0 - \mathrm{F}\,\sigma'' - |h''_0|) + \sum_{i=1}^{s} (p''_i - t''_i - |h''_i| - 1) \\ &= \sum_{i=0}^{s} p''_i - \sum_{i=1}^{s} t''_i - \sum_{i=0}^{s} |h''_i| - \mathrm{F}\,\sigma'' - s. \end{aligned}$$

Hence

$$(T_1 + T_2)\sigma'' = \sum_{i=0}^{s} p''_i + \sum_{i=0}^{s} \operatorname{tot} h''_i - \sum_{i=0}^{s} |h''_i| - \mathrm{F}\,\sigma'' - s. \tag{8.11}$$

To pursue the calculation of $(T_1 + T_2)\sigma''$ we have to take the steps $(4')$ and $(4'')$ in the construction of $\mathbf{D}$, described in Section 5, into account. In case $(4')$,

$$\mathbf{D}\,\sigma' = (\infty 0 h''_0 p''_0)(p''_1 \overline{d''_1} t''_1 h''_1) \cdots (p''_s \overline{d''_s} t''_s h''_s),$$

so, by comparison with (8.9),

$$\begin{aligned}(T_1+T_2)\sigma'' &= \sum_{i=1}^{r} p_i + \sum_{i=0}^{r} \operatorname{tot} h'_i - \left(\sum_{i=0}^{r} |h'_i| - 1\right) - \mathrm{F}\,\sigma'' - r \\ &= \sum_{i=1}^{r} p_i + \sum_{i=0}^{r} \operatorname{tot} h_i - \sum_{i=0}^{r} |h_i| - \mathrm{F}\,\sigma'' - r + 1. \end{aligned} \tag{8.12}$$

In case $(4'')$,

$$\mathbf{D}\,\sigma'' = (\infty 0(h'_0)'')(p''_0((\overline{d'_0})''t''_0(h''_0)''))(p''_1\overline{d''_1}t''_1h''_1)\cdots(p''_s\overline{d''_s}t''_sh''_s).$$

By comparison with (8.9),

$$\begin{aligned}(T_1+T_2)\sigma'' &= \sum_{i=1}^{r} p_i + \sum_{i=0}^{r} \operatorname{tot} h'_i - \sum_{i=0}^{r} |h'_i| - \mathrm{F}\,\sigma' - (r-1) \\ &= \sum_{i=1}^{r} p_i + \sum_{i=0}^{r} \operatorname{tot} h_i - \sum_{i=0}^{r} |h_i| - \mathrm{F}\,\sigma' - r + 1. \end{aligned} \tag{8.13}$$

Thus, the two expressions for $(T_1+T_2)\sigma''$ in (8.12) and (8.13) coincide and

$$(T_1+T_2)\sigma'' - (T_1+T_2)\sigma = \mathrm{F}\,\sigma - \mathrm{F}\,\sigma''. \tag{8.14}$$

As $\operatorname{mak}\mathbf{r} = T_1 + T_2 - (b-ac)$, identity (8.3) will be proved if we show that

$$-(b-ac)\sigma'' + (b-ac)\sigma = \mathrm{L}\,\sigma + \operatorname{des}\sigma'' - n.$$

To do so we make use of the identity

$$(b-ac) = (b-ca) + (ba) + \mathrm{L} - n, \tag{8.15}$$

which is straightforward to prove by induction, and of (1.1), which can be rewritten as

$$(b-ca)\sigma'' = (b-ca)\sigma + n - \mathrm{L}\,\sigma'' - \operatorname{des}\sigma''.$$

Because $(ba)\sigma'' = (ba)\sigma$ by (8.4), we get

$$\begin{aligned}-(b-ac)\sigma'' + (b-ac)\sigma &= -(b-ca)\sigma'' + (b-ca)\sigma - \mathrm{L}\,\sigma'' + \mathrm{L}\,\sigma \\ &= -(b-ca)\sigma - n + \mathrm{L}\,\sigma'' + \operatorname{des}\sigma'' \\ &\quad + (b-ca)\sigma - \mathrm{L}\,\sigma'' + \mathrm{L}\,\sigma \\ &= \mathrm{L}\,\sigma + \operatorname{des}\sigma'' - n. \quad \square\end{aligned}$$

Lemma 8.3. *The two identities hold:* $S\,\Phi = \operatorname{mak}$ *and* $S_1\,\mathbf{r} = S\,\mathbf{r}\,\Phi$.

Proof. We have

$$\begin{aligned} S\,\Phi &= \operatorname{Dbv}\Phi + (b-ca)\,\Phi + (ba)\,\Phi - n + \mathrm{L}\,\Phi && \text{[by (8.3)]}\\ &= \operatorname{Dbv} + (b-ca)\,\Phi + (ba)\,\Phi - n + \mathrm{L}\,\Phi && \text{[by (8.5)]}\\ &= \operatorname{Dbv} + ((b-ca) + n - \mathrm{L}\,\Phi - (ba)\,\Phi) + (ba)\,\Phi - n + \mathrm{L}\,\Phi && \text{[by (1.1)]}\\ &= \operatorname{Dbv} + (b-ca) = \operatorname{mak}. && \text{[by (8.2)]}\end{aligned}$$

On the other hand, the statistic S_1 can be rewritten as

$$S_1 = \mathrm{mak} + \mathrm{F} + \mathrm{L} - n - 1. \tag{8.16}$$

Hence, as $\mathrm{des}\,\mathbf{r} = n - 1 - \mathrm{des}$ and $\mathrm{L}\,\mathbf{r} = \mathrm{F}$, we have

$$\begin{aligned} S\,\mathbf{r}\,\Phi &= \mathrm{mak}\,\mathbf{r}\,\Phi + \mathrm{des}\,\mathbf{r}\,\Phi + \mathrm{L}\,\mathbf{r}\,\Phi - n && \text{[by (8.3)]}\\ &= (\mathrm{mak}\,\mathbf{r} + \mathrm{L} + \mathrm{F} - \mathrm{F}\,\Phi + \mathrm{des}\,\Phi - n)\\ &\quad + (n - 1 - \mathrm{des}\,\Phi) + \mathrm{F}\,\Phi - n && \text{[by (8.8)]}\\ &= \mathrm{mak}\,\mathbf{r} + \mathrm{F}\,\mathbf{r} + \mathrm{L}\,\mathbf{r} - n - 1 = S_1\,\mathbf{r}. \quad \square \end{aligned}$$

It follows from Lemma 8.3 that $S_1\,\mathbf{r}\,\Phi^{-1}\,\mathbf{r} = S$, so $S_1\,\mathbf{r}\,\Phi^{-1}\,\mathbf{r}\,\Phi = S\,\Phi = \mathrm{mak}$. Also $\mathrm{des}\,\mathbf{r}\,\Phi^{-1}\,\mathbf{r}\,\Phi = \mathrm{des}$, by (8.4) and the fact that $\mathbf{r}$ occurs twice in the expression of the transformation $\mathbf{r}\,\Phi^{-1}\,\mathbf{r}\,\Phi$. This achieves the proof of Theorem 8.1.

The proof of Theorem 8.2 is much simpler. It is based upon the straightforward identities

$$\mathrm{des}\,\mathbf{r}\,\mathbf{c} = \mathrm{des} \qquad \text{and} \qquad \mathrm{mak}\,\mathbf{r}\,\mathbf{c} = n\,\mathrm{des} - \mathrm{mak}. \tag{8.17}$$

First,

$$\begin{aligned} \mathrm{des}\,\mathbf{r}\,\mathbf{c}\,\Phi\,\mathbf{r}\,\mathbf{c} &= \mathrm{des}\,\mathbf{r}\,\mathbf{c}\,\Phi && \text{[by (8.17)]}\\ &= \mathrm{des}\,\mathbf{r}\,\mathbf{c} && \text{[by (8.4)]}\\ &= \mathrm{des}\,. && \text{[by (8.17)]} \end{aligned}$$

Then, notice that the statistic S_3 is also equal to

$$S_3 = \mathrm{mak} - \mathrm{des} + \mathrm{F} - 1. \tag{8.18}$$

Hence,

$$\begin{aligned} S_3\,\mathbf{r}\,\mathbf{c}\,\Phi &= \mathrm{mak}\,\mathbf{r}\,\mathbf{c}\,\Phi - \mathrm{des}\,\mathbf{r}\,\mathbf{c}\,\Phi + \mathrm{F}\,\mathbf{r}\,\mathbf{c}\,\Phi - 1 && \text{[by (8.18)]}\\ &= (n\,\mathrm{des}\,\Phi - \mathrm{mak}\,\Phi) - \mathrm{des}\,\Phi + (n + 1 - \mathrm{L}\,\Phi) - 1 && \text{[by (8.17)]}\\ &= n\,\mathrm{des} - (\mathrm{mak}\,\Phi + \mathrm{des}\,\Phi + \mathrm{L}\,\Phi - n)\\ &= n\,\mathrm{des} - S\,\Phi && \text{[by (8.3)]}\\ &= n\,\mathrm{des} - \mathrm{mak} && \text{[by Lemma 8.3]}\\ &= \mathrm{mak}\,\mathbf{r}\,\mathbf{c}. && \text{[by (8.17)]} \end{aligned}$$

Thus $S_3\,\mathbf{r}\,\mathbf{c}\,\Phi\,\mathbf{r}\,\mathbf{c} = \mathrm{mak}$. This achieves the proof of Theorem 8.2.

9. The other statistics S_2 and S_4

Those statistics will be compared with the major index "maj" that reads $\mathrm{maj} = (a - cb) + (b - ca) + (c - ba) + (ba)$. First,

$$S_2 - \mathrm{maj} = (b - ac) - (b - ca) + [b - a) - (ba).$$

As $(b - ac) + (b - a] = (b - ca) + (ba)$, we have $S_2 - \mathrm{maj} = [b - a) - (b - a]$. Also $S_4 - \mathrm{maj} = (b - a) - (ba)$.

With each permutation $\sigma = x_1 x_2 \dots x_n$ associate $\sigma' = x'_1 x'_2 \dots x'_n$, where $x'_i - x_i \equiv n+1-x_n-x_1 \pmod n$. Then, $\sigma \mapsto \sigma'$ is an *involution* for $(x'_i)' \equiv x'_i + n+1-x'_n-x'_1 \equiv (x_i+n+1-x_n-x_1)+n+1-(x_n+n+1-x_n-x_1)-(x_n+n+1-x_n-x_1) \equiv x_i$. Moreover, $\operatorname{des}\sigma' = \operatorname{des}\sigma$ and

$$\begin{aligned} \operatorname{maj}\sigma' &= \operatorname{maj}\sigma + x_1 + x_n - n - 1 \\ &= \operatorname{maj}\sigma + [b-a)\sigma - (b-a]\sigma = S_2\,\sigma. \end{aligned}$$

This shows that $(\operatorname{des}, S_2)$ is equidistributed with $(\operatorname{des}, \operatorname{maj})$.

What about S_4? We have $S_4\,\sigma = \operatorname{maj}\sigma + (b-a)\sigma - (ba)\sigma = \operatorname{maj}\sigma + x_1 - 1 - \operatorname{des}\sigma$. Consider the transformation that maps the permutation $\sigma = x_1 x_2 \dots x_n$ onto $\overline{\sigma} = \overline{x}_1 \overline{x}_2 \dots \overline{x}_n$, where $\overline{x}_n := n+1-x_1$ and

$$\overline{x}_i \equiv x_{i+1} - x_1 \pmod{(n+1)} \qquad (i = 1, 2, \dots, n-1).$$

The map $\sigma \mapsto \sigma'$ is not involutive, but is a bijection of $\mathcal{S}_n$ onto itself and

$$\begin{aligned} &\operatorname{des}\overline{\sigma} = \operatorname{des}\sigma; \\ &\operatorname{maj}\overline{\sigma} = \operatorname{maj}\sigma + x_1 - 1 - \operatorname{des}\sigma = S_4\,\sigma. \end{aligned}$$

Again $(\operatorname{des}, S_4)$ is another bistatistic equidistributed with (des, maj). Thus, Conjecture 11 is correct.

Acknowledgement

The authors thank Einar Steingrímsson for his careful and knowledgeable reading of the paper.

References

[1] Eric Babson, Einar Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, Sém. Lothar. Combin. 44 (B44b) (2000) 18. http://mat.univie.ac.at/~slc/.
[2] Leo Carlitz, q-Bernoulli and Eulerian numbers, Trans. Amer. Math. Soc. 76 (1954) 332–350.
[3] Leo Carlitz, Eulerian numbers and polynomials, Math. Mag. 33 (1959) 247–260.
[4] Pierre Cartier, Dominique Foata, Problèmes combinatoires de permutations et réarrangements, in: Lecture Notes in Math., vol. 85, Springer-Verlag, Berlin, 1969.
[5] Robert J. Clarke, Dominique Foata, Eulerian calculus, II: An extension of Han’s fundamental transformation, European J. Combin. 16 (1995) 221–252.
[6] Gérard Duchamp, Oral communication, 2004.
[7] Dominique Foata, Doron Zeilberger, Denert’s permutation statistic is indeed Euler–Mahonian, Stud. Appl. Math. 83 (1990) 31–59.
[8] Guo-Niu Han, Une transformation fondamentale sur les réarrangements de mots, Adv. Math. 105 (1994) 26–41.
[9] Percy Alexander MacMahon, Combinatory Analysis, vols. 1–2, Cambridge Univ. Press, Cambridge, 1915 (Reprinted by Chelsea, New York, 1955).